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IN  
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by  
Hiroaki ONO

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### Abstract

Effect of inhomogeneity and that of modulation on one-dimensional nonlinear waves in dispersive ( or dissipative ) media are investigated by using nonlinear perturbation methods. To the lowest order of approximation, it is shown for inhomogeneous media that the system of governing equations can be reduced to a simple nonlinear equation which takes a modified form of the Korteweg-de Vries equation with an additional term representing the effect of inhomogeneity. On the other hand, it is shown that the nonlinear modulation also can be described by a simple nonlinear equation which has a form of Schrödinger equation with nonlinear potential.

By using these equations, it is found that the inhomogeneity leads to soliton fission and that the modulation may lead to disintegration of the Stokes wave into envelope solitons. These results are qualitatively in good agreement with those of numerical calculations and of experiments carried out by other authors.



## Chapter I      INTRODUCTION

### \$1-1.              Steady Progressive Waves (SPW)

In recent years, much attention has been paid to the study of a certain class of nonlinear waves such as shock waves in dissipative media and solitary waves in dispersive media. Main reason for this may be due to the fact that such waves have been discovered in many physical systems which have a wide spectrum ranging from classical water layer and lattice to collision-free plasma and electric circuit. Up to the present, various aspects of such waves have been clarified by many authors. Amongst them, a notable result is that solitary waves are strongly stable despite their mutual nonlinear interactions and behave as if they were independent particles. This is the reason why solitary wave is often called 'soliton'. Almost all investigations carried out so far, however, are mainly concerned with waves in homogeneous media and concerned with an elementary process without modulation. The present thesis aims to understand two new aspects of those waves: one is to examine an effect of inhomogeneity and the other the modulation.

Let us first give a brief survey concerning the generation mechanism of such nonlinear waves. It is well known that the nonlinear terms in governing equations of wave motion usually play a role to steepen waveform in the

course of time evolution. If there is no effect preventing such steepening, the wave will eventually break down as frequently observed on beach. In many physical systems, however, there exist usually some sort of smoothing effects. Under such a situation interaction between the two competing effects will govern a long time asymptotic behaviour of the wave, and eventually give rise to the steady state. A typical example of such smoothing effect is the viscous dissipation in viscous fluids and the steady state is known as shock wave.

One of the simple equations which can describe the competition of the nonlinear effect and the smoothing effect may be the Burgers equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad (\nu > 0), \quad (1.1)$$

which was proposed by Burgers as a simple model equation for viscous fluid, and has been used in the study of turbulence<sup>1,2)</sup> and flood waves<sup>3)</sup>.

In this equation the smoothing effect is obviously viscous dissipation. It is easily verified that this equation has a particular solution given by

$$u(x - u_0 t) = u_0 - a_0 \tanh \left\{ \frac{a_0}{2\nu} (x - u_0 t) \right\}, \quad (1.2)$$

which represents a 'shock' wave ( Fig. 1-1) where  $u_0$  and  $a_0$  are integration constants. It is easily seen from the above expression that this shock wave propagates with uniform speed



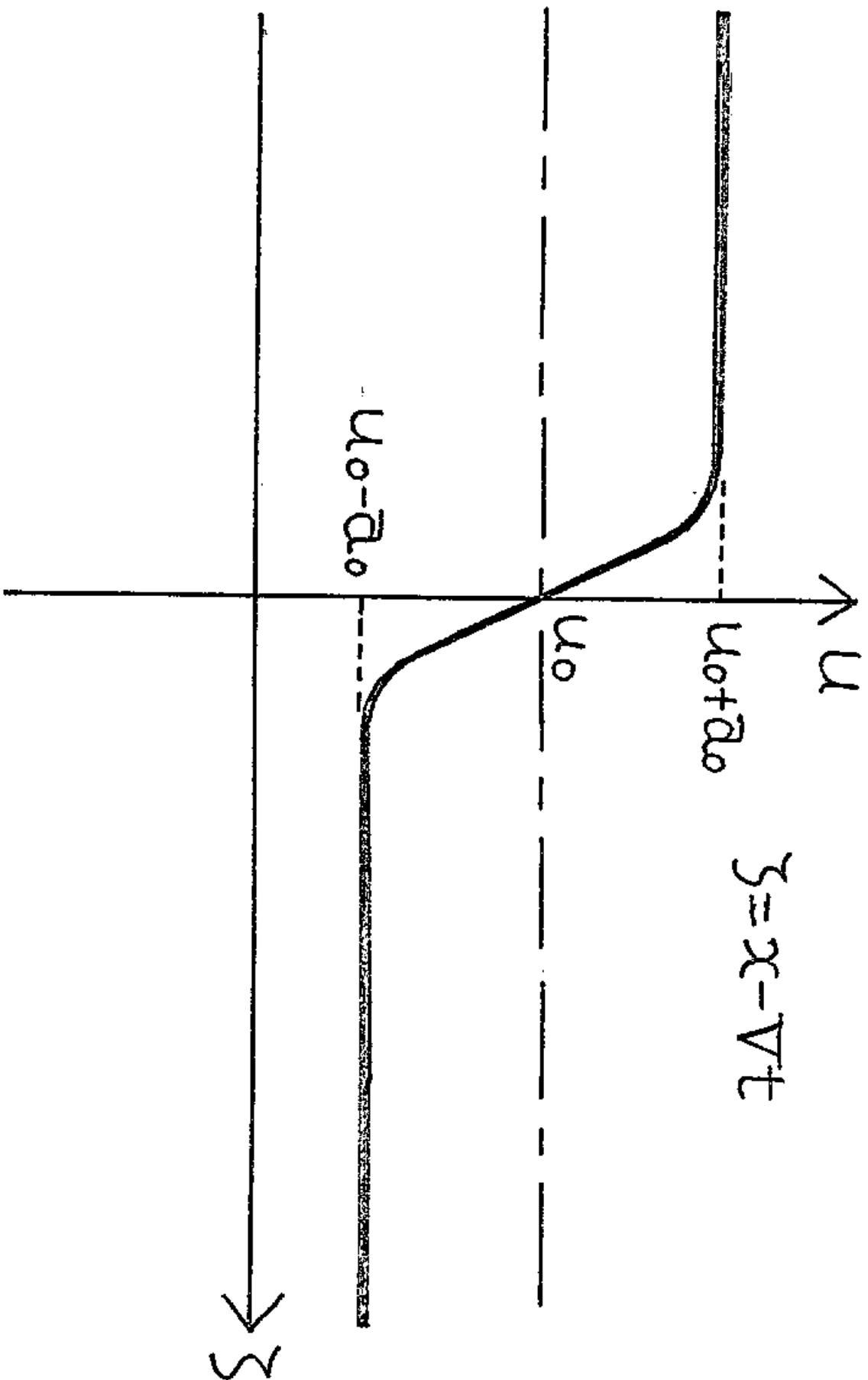


FIGURE 1-1. [ Burgers ] SHOCK WAVE

$u_0$  without change of its waveform. We shall call this sort of steady state 'steady progressive wave (shortly SPW )', which results from the dynamical balance between the effect of nonlinear steepening and that of smoothing. We shall emphasize here that SPW is impossible if either of the two competing effects is absent.

In contrast to the above SPW in dissipative media, various SPW have been found in a wide class of nonlinear dispersive media such as shallow-water layer, collision-free plasma, one-dimensional lattice and electric circuit, where the role of smoothing is played by 'dispersion'. A typical equation describing such a situation, in which the nonlinear effect is smoothed out by the dispersion, may be the Korteweg-de Vries ( K-dV ) equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \mu \frac{\partial^3 u}{\partial x^3} = 0, \quad (1.3)$$

This equation was first obtained for shallow-water waves by Korteweg and de Vries<sup>4)</sup> in 1895. Since then, after a long interval, it has been found in the last decade or so that various nonlinear dispersive waves such as hydromagnetic waves and ion-acoustic waves in collision-free plasma, and one-dimensional lattice waves are also described by this simple nonlinear equation or one of its generalized forms. Physical background together with some mathematical properties of this equation was reviewed in detail by Kakutani<sup>5)</sup> and by Jeffrey and Kakutani<sup>6)</sup>.

The K-dV equation admits the following periodic SPW solutions expressed by the Jacobian elliptic function:

$$u(x - Vt) = u_0 + a_0 \operatorname{sn}^2 \left\{ \sqrt{\frac{a_0}{12\mu}} (x - Vt), s \right\}, \quad (1.4)$$

with

$$V = u_0 + \frac{a_0}{3} \left( 2 - \frac{1}{s^2} \right), \quad (1 \geq s \geq 0),$$

where  $s$  is the modulus of the elliptic function,  $a_0$  and  $u_0$  are integration constants. This periodic SPW is called cnoidal wave because of its functional form (Fig. 1-2(a)). It is quite interesting to note that this periodic wave train degenerates into a solitary wave in the limit of  $s \rightarrow 1$  (Fig. 1-2(b)):

$$u(x - \frac{a_0}{3}t) = a_0 \operatorname{sech}^2 \left\{ \sqrt{\frac{a_0}{12\mu}} \left( x - \frac{a_0}{3}t \right) \right\}, \quad (1.5)$$

which is often called soliton because of its strong stability.

Thus in both dissipative and dispersive media, there exist SPW which are never possible in linear systems. In this sense, SPW seems to play a fundamental role in the mechanics of nonlinear waves. At this stage we should remark that the SPW is not only a steady state solution of system but also an asymptotic solution of the initial value problems. In fact, it has been found numerically and also analytically that a considerably wide class of initial values to the K-dV equation develops into a train of solitons with various scales<sup>7-12</sup>). Further the initial value problems of the Burgers equation reveal that a wide class of initial

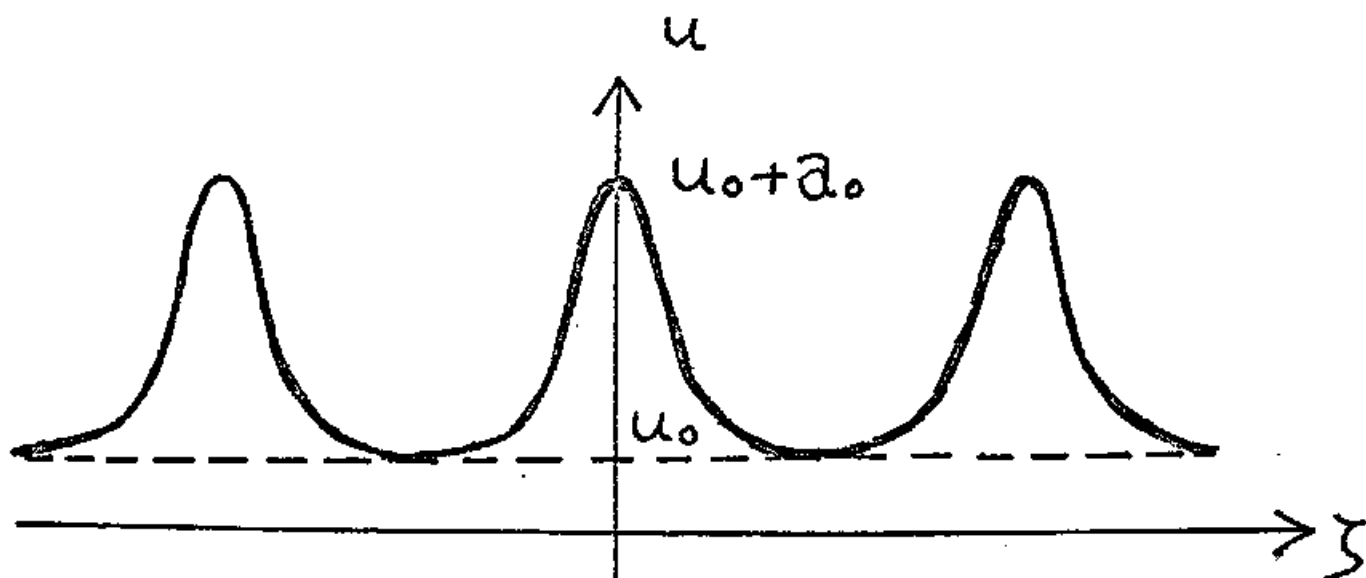


FIGURE 1-2(a).

CNOIDAL WAVE

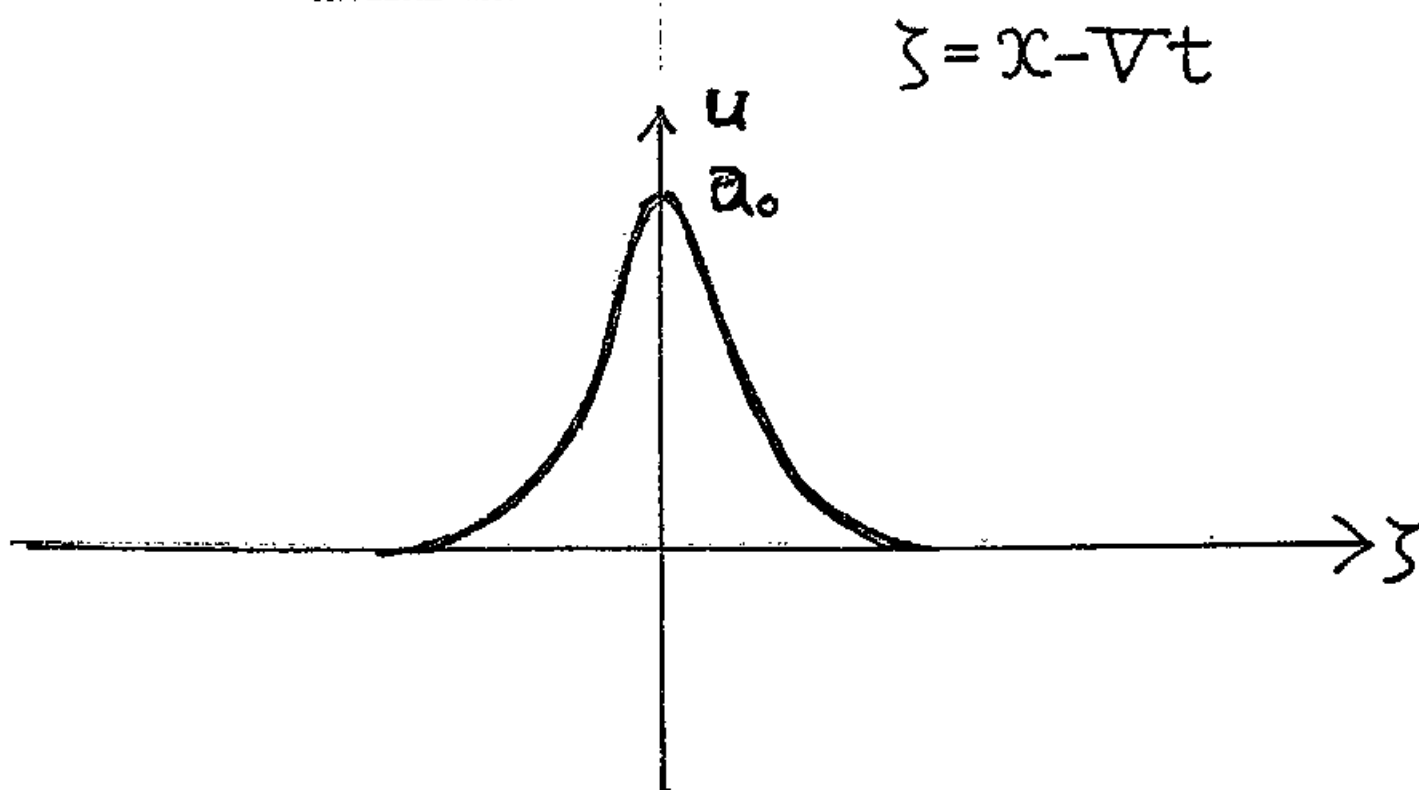


FIGURE 1-2(b).

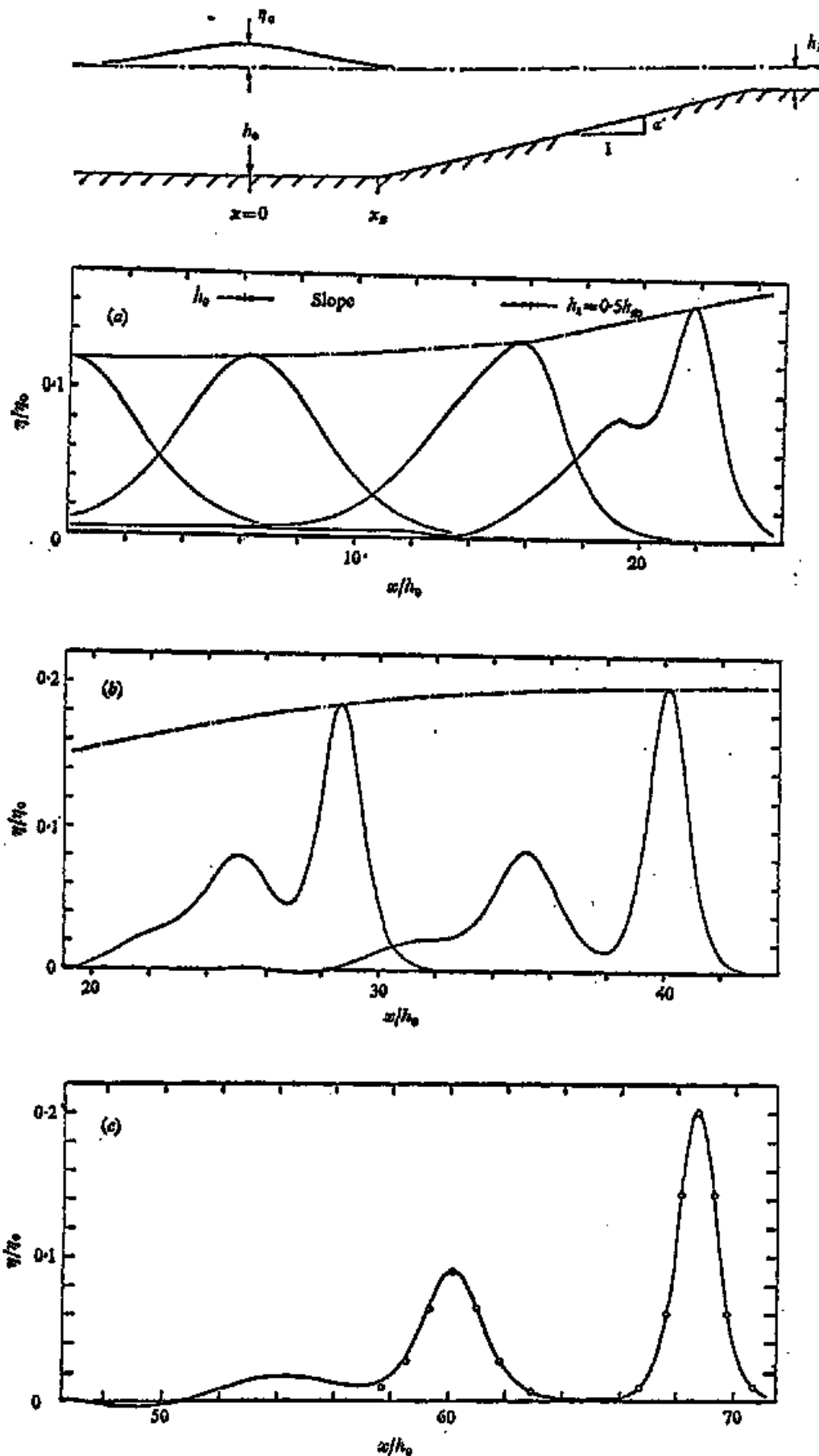
SOLITARY WAVE [ SOLITON ]

disturbances is deformed into a train of saw waves<sup>13)</sup> which is regarded as an array of the Burgers shock waves given by (1.2).

#### §1-2. Effect of Inhomogeneity and Modulation of SPW

Almost all results surveyed in the preceding section are concerned with waves in homogeneous media. In actual physical systems, however, there exist more or less various kinds of inhomogeneity in wave media. Therefore it is not only interesting from academic point of view but also very important for practical applications to examine the effect of inhomogeneity on the wave propagation. This is the reason why we take up this effect as one of the main subjects of the present thesis.

In recent studied of shallow-water waves, the effect of an uneven bottom has been considered by several authors. Madsen and Mei<sup>14)</sup> showed that a solitary wave propagating over a slope onto a shelf splits into a train of solitons ( Fig.1-3). On the other hand, it was shown by Kakutani<sup>15)</sup> and Johnson<sup>16)</sup> that the shallow-water waves over slowly varying uneven bottom can be described by a simple nonlinear equation which is of an extended form of the K-dV equation. Since the K-dV equation governs not only the shallow-water waves but also various other dispersive waves, it may be



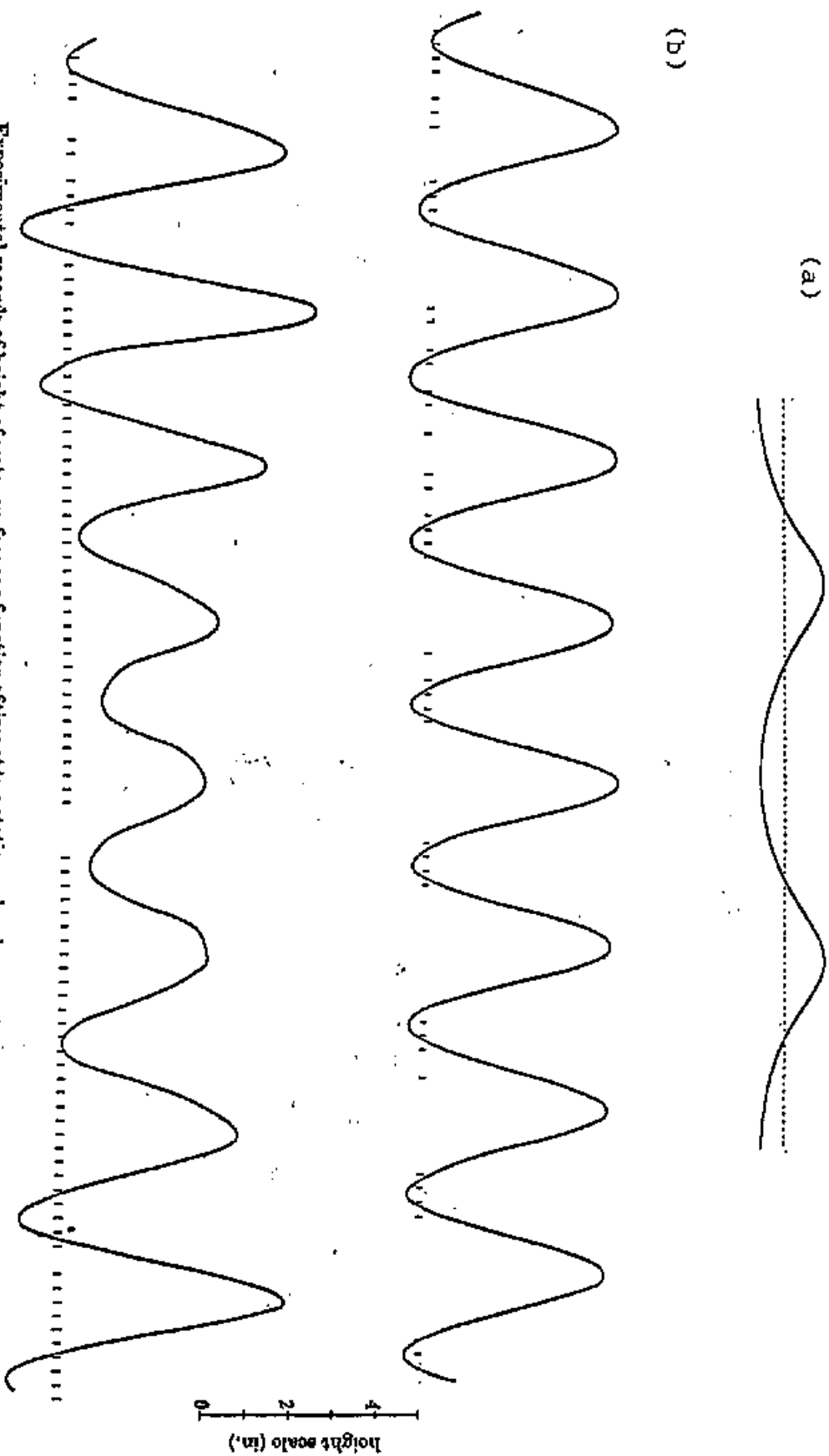
Transformation of a solitary wave ( $\eta_0/h_0 = 0.12$ ) propagating over a slope,  $\alpha = \frac{1}{4}$ , onto a shelf of smaller depth,  $h_1 = 0.5 h_0$ . — indicates the growth of the main crest. In (c), O, indicates theoretical profiles of solitary waves.

FIGURE 1-3. SOLITON FISSION quoted from Madsen and Mei's paper<sup>14)</sup>

expected that waves in considerably wide class of nonlinear dispersive media with inhomogeneity are again governed by this sort of simple equation and that an analogous interesting phenomenon such as soliton fission does occur in other media. Encouraged by this expectation we shall examine in Chap.II an effect of inhomogeneity on lattice waves<sup>17)</sup> and then extend the analysis to more general cases<sup>18)</sup> in Chap.III.

Another subject of this thesis is to investigate the nonlinear modulation of the periodic SPW mentioned in the preceding section. Unlike the isolated pulse such as solitary wave, it is very likely that periodic waves with slightly different wave numbers exchange their energy with each other by nonlinear resonant interaction which may give rise to nonlinear modulation. This kind of resonant instability has been predicted by Benjamin and Feir<sup>19)</sup> for water waves on deep-water layer which are called Stokes waves ( Fig. 1-4). Their analysis, however, is based upon the linear theory and therefore cannot predict a long time evolution of the unstable mode. In Chapter IV, we shall formulate the related problem as a process of nonlinear modulation of the periodic SPW on water layer of arbitrary depth<sup>20)</sup>. A generalization of the results will be made in Chap.V to cnoidal waves in other dispersive media.





Experimental records of height of water surface as a function of time at two stations, showing spontaneous development of instability from background noise. Fundamental frequency, 0.85 c/s. Upper record taken at 200 ft. from wavemaker, lower at 400 ft. Interval between time marks, 0.1 s.

FIGURE 1-4. (a) Stokes Wave (b) Modulational Instability

In general the system of equations governing non-linear waves is usually highly complicated. Even when the basic equation itself is simple as is the case for the water waves, the complexity may be introduced through boundary conditions. In order to avoid the geometrical complexity we restrict our study to the one-dimensional plane waves. Even in this case, it is still very difficult to obtain general explicit analytical results. One of the practical method overcoming such difficulties may be the reduction of original complicated system of equations into a more tractable equation so as to retain the essential features of the original system of equations.<sup>3,21,22</sup>

For this aim, we shall adopt the method of coordinate stretching together with the perturbation expansion of the dependent variables. Namely, on the basis of physical consideration we shall consider the asymptotic behaviour of waves which is determined by the balance between various effects such as nonlinearity, dispersion (or dissipation) and inhomogeneity. We shall first introduce various measures representing each effect. The measure concerned with spacial variation will then be used to stretch the space coordinate. On the other hand, we may use the measure of magnitude of the field quantities as an expansion parameter of the dependent variables. Relative magnitudes of these measures are

chosen so as to couple various effects with each other<sup>18)</sup>.  
Thus, we can obtain an uniformly valid equation which  
governs the asymptotic behaviour of waves.

## Chapter II      Effect of Inhomogeneity on Anharmonic Lattice Waves

### §2-1.      Continuum Model

Let us first consider an effect of inhomogeneity on waves propagating in a one-dimensional anharmonic lattice. It was shown by Zabusky<sup>23)</sup> that the dynamical equation of a one-dimensional anharmonic lossless lattice can be reduced to a simple nonlinear dispersive equation, of which the K-dV equation is a special case, in the limit of a continuum approximation. In his model, Zabusky assumed that the lattice consists of identical particles connected with identical spring constant. In order to clarify an effect of impurity or lattice defect, however, we should take account of some sort of inhomogeneity. It seems also more natural and realistic to suppose that actual molecular systems may be subject to dissipation due to various irreversible processes. It is the aim of this chapter to examine an effect of impurity and that of dissipation on a one-dimensional anharmonic lattice.

The inhomogeneity is introduced by assuming that the mass  $m$  and the spring constant may vary spacially. The dissipative force is assumed to be proportional to the velocity of the constituting particles. Following Zabusky, let us assume that the anharmonic springs, when compressed (or expanded) by an amount  $\Delta$ , exert force given by

$$F = K(\Delta + \alpha \Delta^{p+1}), \quad (2.1)$$

where  $K$  and  $\alpha$ , respectively, are 'linear' spring constant

and a measure of strength of the nonlinearity and  $p$  is a positive constant.

Under these assumptions, the dynamical equation of this lattice ( see Fig. 2.1 ) may be written as:

$$m_i \ddot{y}_i = K_{i+1}(y_{i+1} - y_i) - K_i(y_i - y_{i-1}) + K_{i+1}\alpha(y_{i+1} - y_i)^{p+1} - K_i\alpha(y_i - y_{i-1})^{p+1} - K_i\lambda \dot{y}_i, \quad (2.2)$$

where the subscript  $i$  refers to the  $i$ -th particle,  $y$  is the displacement of the particle, the dot stands for the differentiation with respect to time  $t$ , and  $\lambda$  measures the strength of the dissipation ( assumed to be positive ).

Let us now introduce a continuum model, that is, we assume that the displacement  $y_{i+1}$  can be expressed in terms of the displacement  $y_i$  and its spacial derivatives by means of the Taylor series:

$$y_{i+1} = [y + h y_x + \frac{h^2}{2!} y_{xx} + \frac{h^3}{3!} y_{xxx} + \frac{h^4}{4!} y_{xxxx} + \dots]_i, \quad (2.3)$$

where the subscript  $t$  and  $x$  denote, respectively, the partial differentiation with respect to time  $t$  and space coordinate  $x$ . This approximation may be valid if the inter-particle distance  $h$  is sufficiently small compared with the characteristic length (e.g. wave length) considered here. Similarly we assume that the mass  $m_i$  and the spring constant  $K_i$  are also able to be expanded as follows:

$$\left\{ \begin{matrix} m \\ K \end{matrix} \right\}_{i+1} = \left\{ \begin{matrix} m \\ K \end{matrix} \right\}_i + 0^h \left\{ \begin{matrix} m' \\ K' \end{matrix} \right\}_i + 0^2 (h^2/2!) \left\{ \begin{matrix} m'' \\ K'' \end{matrix} \right\}_i + \dots, \quad (2.4)$$

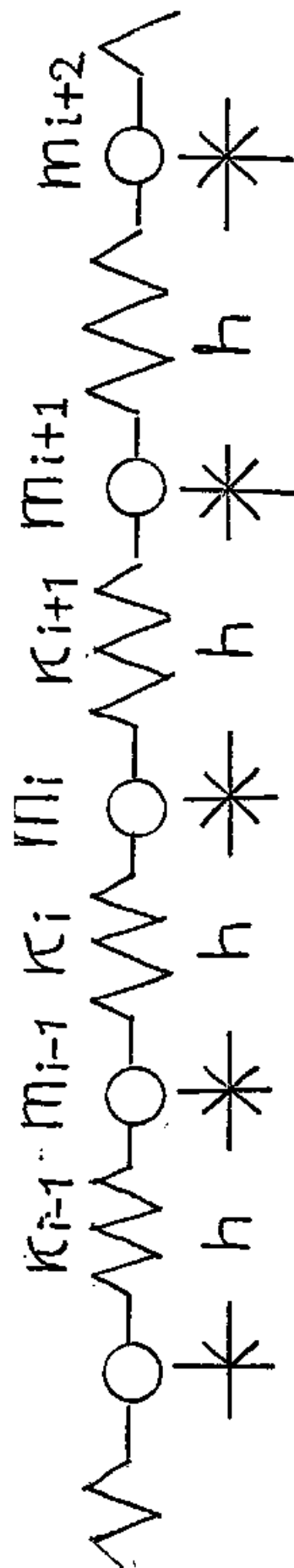


FIGURE 2-1. ONE-DIMENSIONAL LATTICE

where the small parameter  $\sigma$  measures the strength of inhomogeneity, and the prime stands for the differentiation with respect to  $x$ .

Introducing (2.3) and (2.4) into eq. (2.2) and omitting the subscript  $i$ , we obtain

$$y_{tt} = c^2 y_{xx} + c^2 \cdot (p+1) h^p y_x^p y_{xx} + (h^2/12) y_{xxxx} + c^2 \sigma (K/K) y_x - c^2 (Q/h^2) y_t + [\text{higher order terms}] \quad (2.5)$$

where

$$c = \sqrt{h^2 K/m}, \quad (2.6)$$

which may be interpreted as the local propagation speed of the waves in the lowest continuum limit of the linear lossless system, that is, in the limit of  $\sigma \rightarrow 0$  and  $h \rightarrow 0$  leaving  $c$  finite.

## §2-2. Derivation of A Generalized K-d V Equation

In order to reduce Eq. (2.5) into more tractable one, let us introduce the following stretching transformation:

$$\xi = \int \frac{dx}{c} - t, \quad \eta = \xi x, \quad (2.7)$$

where  $\xi$  is a new small parameter representing a measure of dispersion. Introducing (2.7) into Eq. (2.5), setting  $y_\xi = u$  and neglecting the higher order terms, we have:



$$u_{\eta} + A(\eta)u^p u_{\xi} + B(\eta)u_{\xi\xi\xi} + C(\eta)u = 0, \quad (2.8)$$

where

$$\begin{aligned} A(\eta) &= \hat{\alpha}^{(p+1)}/[2c(\eta)^{p+1}], \\ B(\eta) &= 1/[24c(\eta)^3], \\ C(\eta) &= [\frac{d}{d\eta} \ln C(\eta) + \frac{d}{d\eta} \ln m(\eta)]/4 + c(\eta)\hat{\lambda}/4, \end{aligned} \quad (2.9)$$

in which the order of magnitude of the various parameters are chosen in such a way that the effects of nonlinearity, dispersion, inhomogeneity and dissipation balance with each other, that is, we have set

$$\begin{aligned} \alpha &= O(h^{2-p}) = \hat{\alpha}h^{2-p}, \\ \varepsilon &= O(h^2), \\ \sigma &= O(h^2), \\ \lambda &= O(h^4) = \hat{\lambda}h^4, \end{aligned} \quad (2.10)$$

where  $\hat{\alpha}$  and  $\hat{\lambda}$  are assumed to be constants of the order of unity.

We note here that an equation similar to eq.(2.8) was first obtained by Kakutani<sup>11)</sup> in the study of the shallow-water waves over an uneven bottom.

Let us show that eq.(2.8) can be rewritten in a more convenient form. This can be made by setting

$$\psi = \hat{\alpha}[12(p+1)c^{2-p}]^{1/p} u, \text{ and } \tau = \int^{\eta} \frac{d\eta}{24c^3}, \quad (2.11)$$

then eq.(2.8) assumes the following form:

$$\psi_{\tau} + \psi^p \psi_{\xi} + \psi_{\xi\xi\xi} + \nu(\tau)\psi = 0, \quad (2.12)$$

where

$$\nu(\tau) = \frac{1}{4p} \frac{d}{d\tau} \ln(m^{4-p} 3p^{-4}) + 12c^4 \hat{\lambda}, \quad (2.13)$$

the first term of which may be called 'inhomogeneity function'. The second term of (2.13) results from the dissipation (resisting) term and always takes a positive value.

### §2-3. Soliton Fission

In this section we shall consider a typical boundary value problem exemplifying soliton fission by using Eq.(2.12) obtained above. Let us first see how the inhomogeneity affects the total momentum  $Q_1$  and the energy  $Q_2$  of Eq.(2.12), which are constant in homogeneous case, where  $Q_1$  and  $Q_2$  are defined as

$$Q_1 = \int_{-\infty}^{\infty} \psi \, d\xi \quad \text{and} \quad Q_2 = \frac{1}{2} \int_{-\infty}^{\infty} \psi^2 \, d\xi, \quad (2.14)$$

it is assumed here  $\psi$  vanishes  $\xi \rightarrow \pm\infty$ . By using Eq.(2.12) together with the definition of  $Q_1$  and  $Q_2$ , it is easily found that

$$\left. \begin{aligned} Q_1(\tau) &= Q_1(\tau_0) \exp \left\{ - \int_{\tau_0}^{\tau} \gamma(\tau) \, d\tau \right\}, \\ Q_2(\tau) &= Q_2(\tau_0) \exp \left\{ - 2 \int_{\tau_0}^{\tau} \gamma(\tau) \, d\tau \right\}, \end{aligned} \right\} \quad (2.15)$$

where  $\tau$  denotes an appropriate boundary of the region (see Fig. 2.2). It is easily found from eq.(2.15) that the

inhomogeneity leads to decay or growth of waves depending upon the sign of  $\int_{\tau_0}^{\tau} \gamma(\tau) \, d\tau$ . On the other hand, since  $\gamma$  is assumed to be positive, the dissipation effect (the second term (2.13))

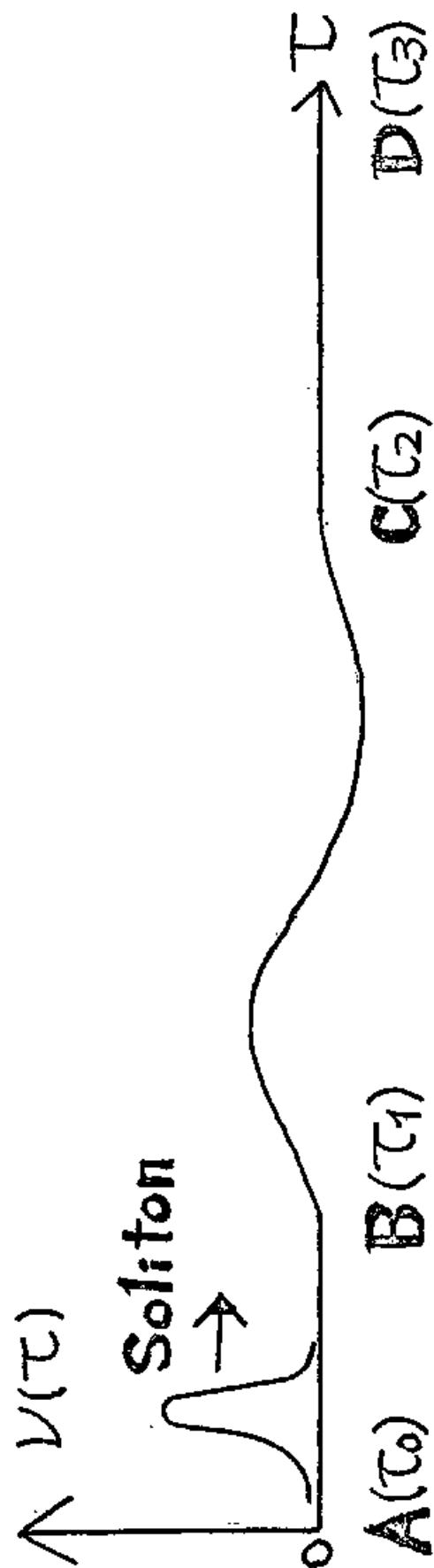


FIGURE 2-2. INHOMOGENEITY FUNCTION

always contributes to decay of waves as expected.

Later we shall see that it is this wave growth that leads to soliton fission.

Using the relations (2.15) between  $V(\tau)$  and  $Q_i$  ( $i=1,2$ ), we shall show how the inhomogeneity affects the behaviour of a soliton which is proceeding from one homogeneous region AB to another homogeneous region CD through inhomogeneous region BC ( see Fig. 2-2 ). From now on we shall deal the equation (2.12) exclusively with  $p=1$  and  $\hat{\lambda}=0$  for simplicity. Hence Eq. (2.12) takes the form:

$$\psi_\tau + \psi\psi_\xi + \psi_{\xi\xi\xi} + V(\tau)\psi = 0, \quad (2.12')$$

where

$$V(\tau) = \frac{1}{4} \frac{d}{d\tau} \ln(m^3 K^{-1}). \quad (2.13')$$

In the homogeneous region AB and CD where  $V(\tau) = 0$ , the equation reduces to the classical K-dV equation. Consequently, when we put the following soliton at the boundary point  $A(\tau_0)$

$$\psi(\tau, \xi) = a_0 \operatorname{sech}^2 \alpha_0 (\xi - \lambda_0 \tau), \quad (2.16)$$

with

$$a_0 / (12\alpha_0^2) = 1 \quad \text{and} \quad \lambda_0 = a_0/3, \quad (2.17)$$

then this soliton will advance towards the entrance B of the inhomogeneous region BC without change in its waveform. But in the region between B and C this is no longer the case.

The soliton will interact there with inhomogeneity  $V(\tau)$  ( $\neq 0$ ) and will be deformed. If the inhomogeneity is sufficiently weak, however, we may suppose that the soliton does not change its waveform very much. Thus we may assume that the solution  $\psi(\xi, \tau)$  takes the following form in the region between B and C:

$$\psi(\tau, \xi) = a(\tau) \operatorname{sech}^2 \alpha(\tau) (\xi - \lambda(\tau) \tau), \quad (2.18)$$

where  $\text{for } \tau_0 \leq \tau \leq \tau_1$  , (2.19)

$$a(\tau_0) / \{12\alpha^2(\tau_0)\} = 1 \text{ and } \lambda(\tau_0) = a(\tau_0)/3 .$$

By virtue of the relations (2.15), the parameters  $a(\tau)$  and  $\alpha(\tau)$  are connected with the inhomogeneity function  $V(\tau)$ , from which we have at the entrance of the new homogeneous region C ( $\tau_1$ )

$$\begin{aligned} a(\tau_1) / \{12\alpha^2(\tau_1)\} &= \exp \left\{ - \int_{\tau_0}^{\tau_1} V(\tau) d\tau \right\} , \\ \alpha(\tau_1) &= \alpha(\tau_0) . \end{aligned} \quad (2.20)$$

Thus the deformed soliton characterized by (2.20) begins to advance through the homogeneous region CD where the wave is again governed by the classical K-dV equation.

On the other hand, for the classical K-dV equation, Gardner et al<sup>9,10)</sup>, Kaepman<sup>11)</sup> and Zabusky<sup>12)</sup> showed that an initial disturbance such as

$$\psi(\tau_1, \xi) = a(\tau_1) \operatorname{sech}^2 \{ \alpha(\tau_1) \xi \} , \quad (2.21)$$

with

$$a(\tau_2)/12\chi^2(\tau_2) = n(n+1)/2, \quad n > 0, \quad (2.22)$$

is disintegrated, for  $\tau > \tau_1$ , into  $n$  solitons or into  $[n]$  solitons accompanied by oscillatory tail according as  $n$  is an integer or not, where  $n > 1$  and  $[ ]$  denotes Gaussian notation. Applying this result to the present situation, we may conclude that the behaviour of the deformed soliton in the new homogeneous region CD is predicted by combining the relations (2.20) and (2.21):

$$n(n+1)/2 = \exp\left\{-\int_{\tau_1}^{\tau_2} \nu(\tau) d\tau\right\}. \quad (2.23)$$

Noting the explicit form of  $\nu(\tau)$  given by (2.13), it is found that a given soliton is disintegrated into a number of solitons when it enters into harder spring region if  $m(\tau)$  is kept constant.

§3-1.      Introduction

In the preceding chapter, it was shown that the effect of inhomogeneity and that of dissipation on the lattice waves can be expressed simply by a single term additional to the equation studied by Zabusky. In this chapter, we shall extend the result to more general cases and show that a considerably wide class of nonlinear dispersive (or dissipative) media with inhomogeneity can also be governed by this type of simple equation.

In homogeneous case, it was found by Taniuti and Wei<sup>21)</sup> that the following class of nonlinear equations are reducible to a simple equation such as K-dV equation, Burgers equation or one of their generalized forms:

$$\frac{\partial U}{\partial t} + A \cdot \frac{\partial U}{\partial x} + \sum_{\beta} \prod_{\alpha}^2 (H_{\alpha}^{\beta} \frac{\partial}{\partial t} + K_{\alpha}^{\beta} \frac{\partial}{\partial x}) U = 0, \quad (3.1)$$

where  $U$  is a column vector with  $n$  components  $u_1, u_2, \dots, u_n$  ( $n \geq 2$ );  $A, H_{\alpha}^{\beta}$ 's and  $K_{\alpha}^{\beta}$ 's are  $n \times n$  matrices whose elements are functions of  $U$  alone, and  $t$  and  $x$  denote the time and space coordinates respectively. It is known that system of equations governing many one-dimensional nonlinear waves can be rewritten in the above matrix form<sup>24)</sup>. The unperturbed steady state of the above system may be usually inhomogeneous in itself or because of certain kinds of external boundary conditions.



We shall consider here nonlinear waves propagating through such inhomogeneous media<sup>18)</sup>. It is shown, to the lowest order of perturbation around the inhomogeneous state, that the system of governing equations can be reduced to a single nonlinear equation of the type similar to Eq.(2.12).

### §3-2 Reduction of The System of Governing Equations

Let us now consider the following equation for a column vector  $U$  with  $n$  components  $u_1, u_2, \dots, u_n$ ; ( $n \geq 2$ ):

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} + \sum_{\beta} \prod_{\alpha}^2 (H_{\alpha}^{\beta} \frac{\partial}{\partial t} + K_{\alpha}^{\beta} \frac{\partial}{\partial x}) U + B \frac{dS}{dx} = 0, \quad (3.2)$$

where  $S$  is a vector valued function of  $x$ ;  $A$ ,  $B$ ,  $H_{\alpha}^{\beta}$ 's and  $K_{\alpha}^{\beta}$ 's are  $n \times n$  matrices with elements depending upon  $U$  and  $S$ . For many physical systems, the inhomogeneity due to the external boundary conditions can generally be expressed by the last term in eq.(3.2). The steady state of the system  $U_0$  is given by

$$A_0 \frac{dU_0}{dx} + \sum_{\beta} \prod_{\alpha}^2 (K_{0\alpha}^{\beta} \frac{d}{dx}) U_0 + B_0 \frac{dS}{dx} = 0, \quad (3.3)$$

where  $A_0$ ,  $B_0$  and  $K_{0\alpha}^{\beta}$ 's are the respective values of  $A$ ,  $B$  and  $K_{\alpha}^{\beta}$ 's for  $U=U_0$ .

In order to examine the coupling among the effects of nonlinearity, dispersion (or dissipation) and inhomogeneity, we first introduce the stretching transformation similar

to (2.7):

$$\xi = \xi^a \left( \int \frac{dx}{c_0} - t \right) \quad \text{and} \quad \eta = \xi^{a+1} x, \quad (3.4)$$

with  $a = 1/(q-1)$ ,

where  $c_0$  is a non-degenerate real eigen value of  $A_0 = A(U_0)$  and represents the propagation speed of the linear wave, and  $\xi$  measures the magnitude of nonlinearity.

In view of the discussion given in §1-3,  $\xi^a$  is regarded as a measure of dispersion (or dissipation). Further we assume here that  $U_0$  and  $S$  are slowly varying functions of  $x$  which depend on  $\eta$  alone. Therefore  $\xi^{a+1}$  is interpreted as a measure of inhomogeneity.

We now expand  $U$  around  $U_0$  in power of  $\xi$  as

$$U = U_0 + \xi U_1 + \xi^2 U_2 + \dots, \quad (3.5)$$

then corresponding to this,  $A$ ,  $B$ ,  $H_\alpha^\beta$ 's, and  $K_\alpha^\beta$ 's can also be expanded as follows:

$$\begin{aligned} A &= A_0 + \xi A_1 + \dots, \\ B &= B_0 + \xi B_1 + \dots, \\ H_\alpha^\beta &= H_{0\alpha}^\beta + \xi H_{1\alpha}^\beta + \dots, \\ \text{and} \\ K_\alpha^\beta &= K_{0\alpha}^\beta + \xi K_{1\alpha}^\beta + \dots, \end{aligned} \quad (3.6)$$

Introducing these expressions (3.4)-(3.6) into eq.(3.2) and using the equation (3.3), we can derive, after some manipulations, the following equation:

$$\mathcal{P}_\eta + \alpha(\eta) \mathcal{P} \mathcal{P}_\xi + \beta(\eta) \underbrace{\mathcal{P}_\xi \dots \mathcal{P}_\xi}_q + \gamma(\eta) \mathcal{P} = 0, \quad (3.7)$$

where  $\varphi$  is defined by

$$U_1 = r_0 \varphi, \quad (3.8)$$

and the coefficients are given by

$$\begin{aligned} \alpha &= l_0 r_0 \cdot (\nabla_u A)_0 r_0 / (c_0^2 l_0 r_0), \\ \beta &= l_0 \sum_{\alpha} \prod_{\alpha}^2 (-H_{\alpha\alpha} \beta + \frac{1}{c_0} K_{\alpha\alpha} \beta) r_0 / (c_0 r_0 l_0), \\ \gamma &= \{ c_0 l_0 r_0 \eta + l_0 r_0 [(\nabla_u A)_0 U_0 \eta + (\nabla_u B)_0 S \eta] \} / (c_0 r_0 l_0) \end{aligned} \quad (3.9)$$

in which  $r_0$  and  $l_0$  are, respectively, the right and the left eigen vector of  $A_0$  associated with the eigen value  $c_0$ , and use is made of the relations

$$A_1 = U_1 \cdot (\nabla_u A)_0 = \sum_{i=1}^n u_{1i} (\partial A / \partial u_i)_{U=U_0},$$

and likewise for  $B_1$ .

It should be noted here that when  $\beta = 0$ , there is no interaction between the nonlinearity and dissipation (or dispersion) effect in this order of approximation. Consequently, as stated in Chapt. I, the differentiability of the solution to eq. (3.7) with  $\beta = 0$ , is not necessarily guaranteed for long time, although this does not mean the invalidity of the original system of equations (3.2). In this case, which we may call 'exceptional' case in contrast to 'general' case where  $\beta \neq 0$ , we must make re-ordering of the independent variables so as to include the effect of dispersion (or dissipation). To achieve this aim, we use the stretching transformation (3.4) with replacing  $a = 1/(q-1)$  by  $a = 1/2(q-1)$ . Furthermore we replace the expansion (3.5) by

$$U = U_0 + \xi^{3/2} U_2 + \xi^2 U_3 + \dots, \quad (3.10)$$

and likewise for  $A$ ,  $B$ ,  $H_\alpha^\beta$ 's and  $K_\alpha^\beta$ 's. After similar calculations to those for the general case, we can obtain the following equation instead of eq.(3.7):

$$\varphi_\eta + \alpha'(\eta) \varphi_\xi + \beta'(\eta) \varphi_{\xi \dots \xi} + \gamma'(\eta) \varphi = 0, \quad (3.11)$$

$\begin{matrix} \xi \\ \vdots \\ \xi \\ \hline 2q-1 \end{matrix}$

where the coefficients are given by

$$\begin{aligned} \alpha' &= \alpha, \\ \beta' &= 1_0 \sum_{\beta} \prod_{\alpha}^{\xi} (-H_{0\alpha}^\beta + \frac{1}{c_0} K_{0\alpha}^\beta) s_0 / (c_0 1_0 r_0), \\ \gamma' &= \gamma, \end{aligned} \quad (3.12)$$

in which  $s_0$  is the column vector satisfying the equation:

$$(-I + \frac{A_0}{c_0}) s_0 = - \sum_{\beta} \prod_{\alpha}^{\xi} (-H_{0\alpha}^\beta + \frac{1}{c_0} K_{0\alpha}^\beta) r_0. \quad (3.13)$$

By virtue of a similar transformation to (2.11), both of the equations (3.7) and (3.11) can be rewritten in a form similar to that of Eq.(2.12). In fact, setting

$$\psi = \begin{Bmatrix} \alpha/\beta \\ \alpha'/\beta' \end{Bmatrix} \quad \text{and} \quad \tau = \int \begin{Bmatrix} \beta \\ \beta' \end{Bmatrix} d\eta, \quad (3.14)$$

we have:

$$\psi_\tau + \psi_\psi + \psi_{\xi \dots \xi} + \gamma(\tau) \psi = 0, \quad (3.15)$$

$\begin{matrix} \xi \\ \vdots \\ \xi \\ \hline \begin{matrix} q \\ 2q-1 \end{matrix} \end{matrix}$

where the inhomogeneity functions  $\mathcal{V}(\tau)$  are given by

$$\mathcal{V}(\tau) = \left\{ \begin{array}{l} \gamma/\beta \\ \gamma'/\beta' \end{array} \right\} + \frac{d}{d\tau} \ln \left\{ \begin{array}{l} \beta/\alpha \\ \beta'/\alpha' \end{array} \right\} \quad (3.16)$$

In the above equations the upper and the lower lines corresponding, respectively, to the general case ( $\beta \neq 0$ ) and to the exceptional case ( $\beta = 0$ ).

### §3-3. Physical Examples

Several physical examples which are governed by Eqs. (3.15) are presented in this section. They are weak shock waves in a duct with varying cross section, shallow-water waves over an uneven bottom and oblique magneto-acoustic waves in a collision-free plasma with inhomogeneous distribution of plasma density. In the homogeneous case, the first example is governed by the Burgers equation, while the remaining two by the K-dV equation. The inhomogeneity function for each medium is listed up in Table-I.

Thus it is found that the equation of the form (3.15) has a wide applicability as its homogeneous analogue does. In view of this, it is highly plausible that remarkable phenomena such as soliton fission may occur in various inhomogeneous media.

TABLE-I INHOMOGENEITY FUNCTIONS

MEDIUM	FLOW IN DUCT	LATTIC	PLASMA	WATER LAYER
q	$2 (\beta \neq 0)$	3	$2 (\beta \neq 0)$	$3 (\beta \neq 0)$
$\nu(\tau)$	$\frac{1}{2} \frac{d}{d\tau} \left( \ln a_0 \frac{d}{d\tau} \right) + \frac{M_0 \tau}{M_0 + 1}$	$-\frac{1}{4} \frac{d}{d\tau} \left( \frac{d}{d\tau} \ln K m^3 \right)$	$-\frac{1}{4} \frac{d}{d\tau} \left( \frac{d}{d\tau} \ln n \right)$	$-\frac{9}{4} \frac{d}{d\tau} \left( \frac{d}{d\tau} \ln h \right)$

$\rho_0$ : DENSITY       $K$ : SPRING CONSTANT       $n$ : DENSITY       $h$ : DEPTH OF  
 $a_0$ : SOUND SPEED       $m_0$ : PARTICLE MASS      STILL WATER  
 $M_0$ : Mach NUMBER

Let us now apply the formula (2.23) for the soliton fission to the shallow-water wave problem which is an only example investigated experimentally. As already mentioned, soliton fission in inhomogeneous media was discovered experimentally by Madsen and Mei<sup>14)</sup> for shallow-water. Substituting the inhomogeneity function  $\gamma(\tau)$  for the shallow-water waves in Table-I into the formula (2.23), we have:

$$h(\tau_1)/h(\tau_2) = [n(n+1)/2]^{4/9}, \quad (3.17)$$

which was also obtained by Johnson<sup>16)</sup>. This suggests that when a soliton enters into shallower region ( $h(\tau_1) > h(\tau_2)$ ), it is disintegrated into  $[n]$  solitons since  $n > 1$ .

In particular, when  $h(\tau_1)/h(\tau_2) = 1/2$ , which is the case studied by Madsen and Mei,  $n$  becomes nearly equal to 2.7. Consequently two solitons accompanied by oscillatory tail must appear, which is in good agreement with the experimental result ( see Fig. 1-3).



§4-1.                      Introduction

In this chapter, we shall consider the nonlinear modulation of the 'periodic' SPW such as the cnoidal wave solution to the K-dV equation.

The stability or modulation of the water wave trains has been investigated by Benjamin and Feir<sup>19)</sup>, Benjamin<sup>25)</sup> and Whitham<sup>26)</sup>. They showed that the Stokes waves on deep-water layer are unstable against a certain class of small disturbances. Since their analysis, however, is based upon the linear theory, they cannot answer the final evolution of such instability. On the other hand, recently Chu and Mei<sup>27)</sup> derived, by modifying Whitham's system of equations so as to include a dispersion effect of envelope waves, a system of equations which can describe a nonlinear behaviour of the unstable disturbances. Their results, however, are limited to the case of infinite depth.

In this chapter, we consider general modulation of water waves which takes place on water layer with arbitrary depth<sup>20)</sup>. It is found by using a nonlinear perturbation method that slow modulation of such waves can be described by a simple nonlinear Schrödinger equation. Using this simple equation, we can reproduce almost all essential results so far obtained for Stokes waves. Moreover we can extend the results to the case of shallow-water-waves.

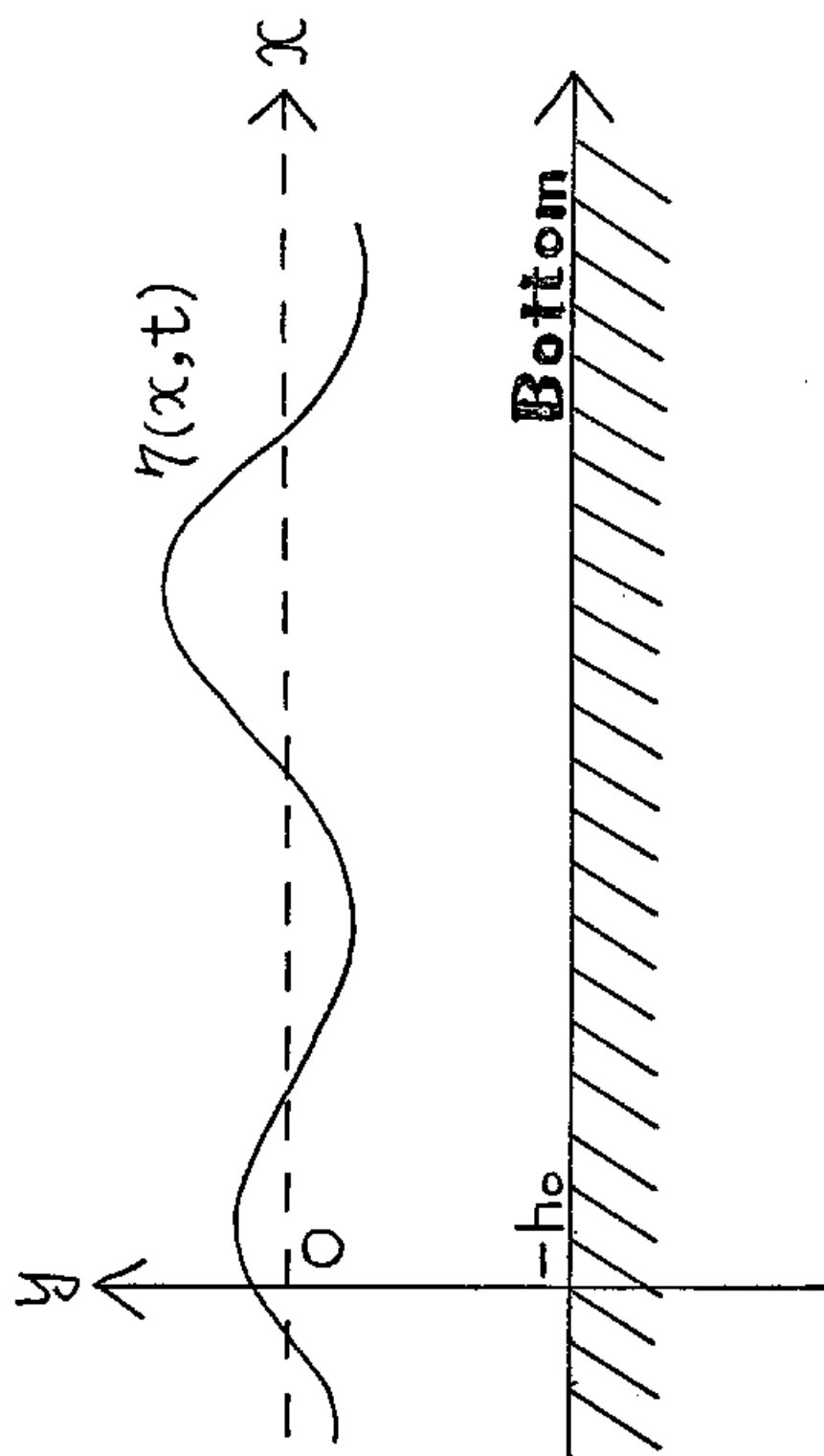


FIGURE 4-1. WATER WAVE

#### §4-2. Derivation of Nonlinear Schrodinger Equation

The basic equation governing water waves, which take place on the free surface of a two-dimensional horizontal layer of a perfect incompressible fluid with an arbitrary depth  $h_0$  is simply expressed as a harmonic equation for the velocity potential  $\phi(x, y, t)$ :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \text{for } -h_0 < y < \eta(x, t), \quad (4.1)$$

with boundary conditions:

$$\frac{\partial \phi}{\partial y} = 0, \quad \text{at } y = -h_0, \quad (4.2)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} \quad \text{at } y = \eta(x, t), \quad (4.3)$$

and

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] + g\eta = 0, \quad \text{at } y = \eta(x, t), \quad (4.4)$$

where  $x$ ,  $y$  and  $t$  denote, respectively, the horizontal, vertical and time coordinates,  $\eta(x, t)$  the perturbed surface measured from the still surface  $y = 0$ , and  $g$  the gravity constant.

Linearizing the system of equations (4.1) - (4.4) and assuming a sinusoidal wave proportional to  $\exp\{i(kx - \omega t)\}$ , we have the well-known linear dispersion relation:

$$\omega^2 = kg \tanh kh_0, \quad (4.5)$$

where  $\omega$  and  $k$  are the frequency and the wave number of the linear waves, respectively.

Let us now consider slow nonlinear modulation of a

wave train, whose frequency and wave number  $\omega_0$  and  $k_0$ , respectively, in the linearized limit. The fundamental frequency  $\omega_0$  and wave number  $k_0$  are assumed to be of the order of unity. In order to obtain uniformly valid equation, it is convenient to introduce the following coordinate stretching transformation:

$$\xi = \varepsilon(x - \lambda_0 t) \quad \text{and} \quad \tau = \varepsilon^2 t, \quad (4.6)$$

where  $\varepsilon$  is a small parameter measuring a weakness of dispersion and  $\lambda_0 (= \omega_0 / k_0)$  is the group velocity of the linear wave.

On the other hand, since we consider weak nonlinear modulation, the dependent variables are assumed to be expanded in terms of  $\varepsilon$ . In this sense  $\varepsilon$  is also regarded as a measure of nonlinearity.

$$\phi(x, y, t) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^n \phi^{(n,m)}(\xi, y, \tau) e^{im(k_0 x - \omega_0 t)}, \quad (4.7)$$

$$\eta(x, t) = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-n}^n \eta^{(n,m)}(\xi, \tau) e^{im(k_0 x - \omega_0 t)}, \quad (4.8)$$

where  $\phi^{(n,m)}$  and  $\eta^{(n,m)}$  are complex amplitudes representing the modulation of amplitude and phase of the fundamental wave train. Since  $\phi$  and  $\eta$  are real functions,  $\phi^{(n,m)*} = \phi^{(n,-m)}$  and  $\eta^{(n,m)*} = \eta^{(n,-m)}$  must be satisfied, where the asterisk denotes the complex conjugate.

Introducing these expressions (4.6)-(4.8) into the system of equations (4.1)-(4.4), we can derive, after some manipulations, the following equation:

$$i \frac{\partial \phi^{(1,1)}}{\partial \tau} + \mu \frac{\partial^2 \phi^{(1,1)}}{\partial \xi^2} + \nu |\phi^{(1,1)}|^2 \phi^{(1,1)} = 0, \quad (4.9)$$

where  $\psi^{(1,1)}$  is defined as

$$\psi^{(1,1)}(\xi, \tau) = \phi^{(1,1)} / \left[ \frac{\cosh k_0(y+h_0)}{\cosh k_0 h_0} \right], \quad (4.10)$$

and the coefficients are given by

$$\mu = \frac{1}{2} \frac{\partial \omega_0}{\partial k_0^2} \quad (4.11)$$

and

$$\nu = -\frac{k_0^4}{4\omega_0} \left( \frac{g \tanh^4 k_0 h_0 - 10 \tanh^2 k_0 h_0 + 9}{2 \tanh^2 k_0 h_0} + \frac{(\tanh^2 k_0 h_0 - 1)^2 g h_0 - 4 (\tanh^2 k_0 h_0) \lambda_0^2 / k_0^2 + 4 \frac{\omega_0^2}{k_0^2}}{\lambda_0^2 - g h_0} \right) \quad (4.12)$$

On the other hand, the lowest order surface elevation can be expressed in terms of  $\psi^{(1,1)}$  as follows:

$$\begin{aligned} \eta^{(1,0)} &= 0, \quad \eta^{(1,1)} = i \gamma_1 \psi^{(1,1)}, \quad \eta^{(2,0)} = \gamma_2 |\psi^{(1,1)}|^2, \\ \eta^{(2,1)} &= \gamma_3 \psi^{(1,1)}, \quad \eta^{(2,2)} = \gamma_4 \psi^{(1,1)^2}, \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} \gamma_1 &= \omega_0 / g, \quad \gamma_2 = \frac{2 \lambda_0 \omega_0 k_0 - g h_0 (\omega_0^2 / g^2 - k_0^2)}{\lambda_0^2 - g h_0}, \\ \gamma_3 &= \lambda_0 / g, \quad \gamma_4 = \frac{k_0^3 (\tanh^2 k_0 h_0 - 3)}{2 \omega_0^2 \tanh k_0 h_0}. \end{aligned} \quad (4.14)$$

It is found from (4.11) and (4.12) that  $\mu$  and  $\nu$  are real

functions of the unperturbed depth  $h_0$ , and that  $\mu$  always takes negative sign, whereas  $\nu$  changes its sign from negative to positive across  $k_0 h_0 = 1.363$  as  $k_0 h_0$  decreases.

It should be noted that  $-\nu$  is essentially identical with  $X(K)$  defined by his eq.(30) in Benjamin's paper<sup>25)</sup>. It should be remarked here that the above type of equation, which may be called nonlinear Schrödinger equation, has also been obtained for describing the nonlinear modulation in various media<sup>27,28)</sup>

If, instead of the complex amplitude  $\psi^{(1,1)}$ , we use the pair of real functions  $A$  and  $\Omega$  defined by:

$$\psi^{(1,1)} = A \exp \left\{ i \int \frac{\Omega}{2\mu} d\xi \right\}, \quad (4.15)$$

we obtain the following set of equations:

$$\frac{\partial A^2}{\partial \tau} + \frac{\partial (A^2 \Omega)}{\partial \xi} = 0, \quad (4.16)$$

$$\frac{\partial \Omega}{\partial \tau} + \Omega \frac{\partial \Omega}{\partial \xi} - 2\mu \frac{\partial A^2}{\partial \xi} - 2\mu^2 \frac{\partial}{\partial \xi} \left( \frac{1}{A} \frac{\partial A^2}{\partial \xi} \right) = 0, \quad (4.17)$$

which reduces, in the limit of  $k_0 h_0 \rightarrow \infty$  (i.e. infinite depth), to the set of equation adopted by Chu and Mei<sup>27)</sup>.

#### §4-3. Results

Let us now give attention to various particular solutions.

##### i) Basic SPW solution

It is known that Eq.(4.9) has the following solution representing a nonlinear plane wave<sup>29)</sup>:

$$\psi^{(1,1)} = \psi_0 \exp(i\alpha_0 \tau), \quad (4.18)$$

$$\text{where } \psi_0 = \text{constant and } \alpha_0 = \mu |\psi_0|^2. \quad (4.19)$$

In order to clarify the physical meaning of this solution, we first put:

$$\psi_0 = a/(2i\chi_1), \quad a: \text{real} \quad (4.20)$$

thereby  $a$  represents the amplitude of the surface elevation. Substituting (4.18) together with (4.19) and (4.20) into (4.8), we have the following series in  $\epsilon$ :

$$\eta = \epsilon a \cos \zeta + \epsilon^2 \left[ -\frac{Y_4}{4Y_1} \cos 2\zeta + \frac{Y_2}{4Y_1^2} \right] + O(\epsilon^3), \quad (4.21)$$

with

$$\zeta = k_0 x - (\omega_0 - \epsilon^2 \alpha_0) t. \quad (4.22)$$

In the deep-water limit ( $k_0 h_0 \gg 1$ ), this is nothing but the well known Stokes wave to the second order approximation (Fig.1-4). On the other hand, in the shallow-water limit ( $k_0 h_0 \ll 1$ ), this gives the first two terms of the series expansion of the cnoidal wave (1.4) with respect to the small modulus, provided that we define  $k_0$  as

$$k_0^2 = \frac{3}{2} \frac{a \epsilon}{s^2 h_0} \frac{\pi^2}{K^2}, \quad (4.23)$$

where  $K$  is the first kind of complete integral.

In this sense, the solution (4.18) in the limit of shallow-water may be called 'weak' cnoidal wave (Fig.1-2(a)).

## ii) Stability of the basic SPW

In this subsection, let us consider the stability of the basic SPW obtained above. It is found that small modulational disturbances superposed on the basic SPW are governed by the following dispersion relation:

$$\hat{\omega}^2 = \mu^2 \hat{k}^2 (\hat{k}^2 - 2\nu/\mu |\psi_0|^2). \quad (4.24)$$

where the disturbance has been decomposed into Fourier component proportional to  $\exp\{i(\hat{k}\xi - \hat{\omega}\tau)\}$ . It follows from this relation that  $\hat{\omega}$  is real for  $\mu < 0$  while  $\hat{\omega}$  is imaginary for  $\mu > 0$  and  $\hat{k} = \sqrt{2\nu/\mu} |k_0|$ .

Noting the discussion concerning the signs of  $\mu$  and  $\nu$  given in §4-2, we find that the basic periodic SPW on sufficiently deep-water layer (i.e. Stokes wave) is unstable. On the other hand, the basic periodic SPW on sufficiently shallow-water layer (i.e. weak cnoidal wave) is neutrally stable.

The maximum growth rate of the unstable mode  $\delta_{\max}$  is given by

$$\delta_{\max} = |\nu k_0| \quad \text{for} \quad \hat{k} = \sqrt{\nu/\mu} |k_0|, \quad (4.25)$$

These results reproduce the criterion of the stability of the water waves studied by Benjamin<sup>25)</sup> and Whitham<sup>26)</sup>.

The fact that the weak cnoidal wave is stable may be explained if we note that the second term in  $\nu$  (see (4.12)) contributes to the sign change of  $\nu$  at  $k_0 h_0 = 1.363$ , and that this term has its origin in nonlinearly induced mean depth  $\eta^{(2,0)}$  that vanishes in the limit of infinite depth ( $k_0 h_0 \rightarrow \infty$ ).

### iii) New SPW solutions

In order to clarify the final evolution of the unstable modes discussed above, we now return to the original nonlinear Schrodinger equation (4.9). We first note that this equation has another type of solution which represents the dynamical balance between the effects of nonlinearity and of dispersion.



This is expressed by ,                      lliptic function( Fig.4-2(a)):

$$\psi^{(1,1)} = A_0 \operatorname{dn} \left\{ A_0 \sqrt{\frac{\nu}{2\mu}} \xi, s \right\} e^{i\alpha_0 \tau} \text{ for } \mu\nu > 0, \quad (4.26)$$

where the modulus  $s$  is given by

$$s^2 = 2 - \frac{2\alpha_0}{\nu A_0^2}, \quad (1 \leq s \leq 1) \quad (4.27)$$

and  $\alpha_0$  is constant. If, in particular,  $s=1$ , this takes the form of solitary wave with width  $\sqrt{\frac{\mu}{\alpha_0}}$ :

$$\psi^{(1,1)} = \sqrt{\frac{2\alpha_0}{\nu}} \operatorname{sech} \left\{ \sqrt{\frac{\alpha_0}{\mu}} \xi \right\} e^{i\alpha_0 \tau}, \quad (4.28)$$

which may be called 'envelope' soliton (shortly E-soliton ) in contrast to the conventional soliton (1.5) of the K-dV equation ( Fig. 4-2(b)). Thus we have a new type of SPW.

We now notice that the width of the E-soliton is exactly identical with the wave length of the most unstable mode ( cf. (4.25)). In view of this, we can make a conjecture that the unstable Stokes wave is eventually deformed into E-solitons. In fact, this conjecture is strongly supported by many authors<sup>27,30 31</sup>).

On the other hand, it is highly probable that if the linearly stable cnoidal waves are nonlinearly modulated, they may be deformed into the following envelope SPW ( Fig. 4-3(a)):

$$\psi^{(1,1)} = A_0 \operatorname{sn} \left\{ \frac{A}{s} \sqrt{-\frac{\nu}{2\mu}} \xi, s \right\} e^{i\alpha_0 \tau} \text{ for } \mu\nu < 0, \quad (4.29)$$

with the modulus

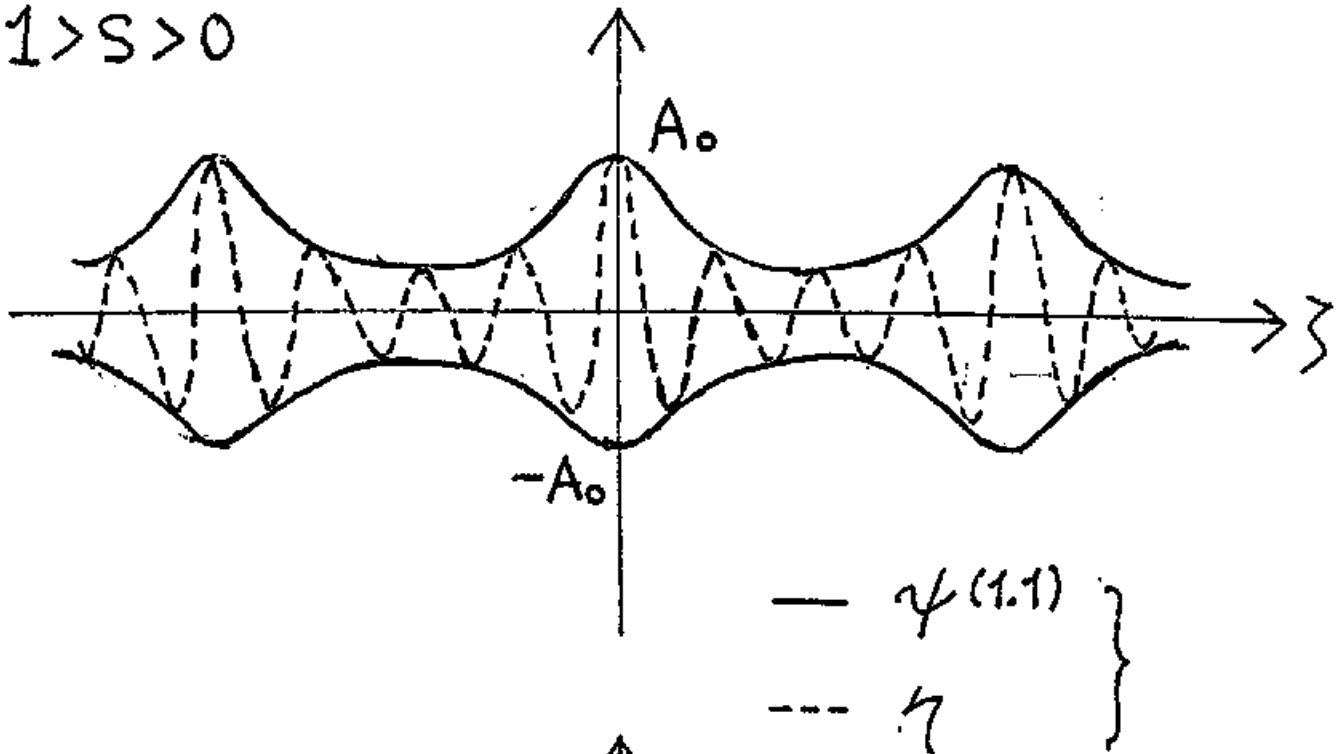
$$s^2 = A_0^2 / \left( \frac{2\alpha_0}{\nu} - A_0^2 \right). \quad (4.30)$$

FIGURE 4-2(a).

PERIODIC ENVELOPE SPW ( $\mu\nu > 0$ )

$$\zeta = A_0 \sqrt{\frac{\nu}{2\mu}} \xi$$

$$1 > S > 0$$



$$S = 1$$

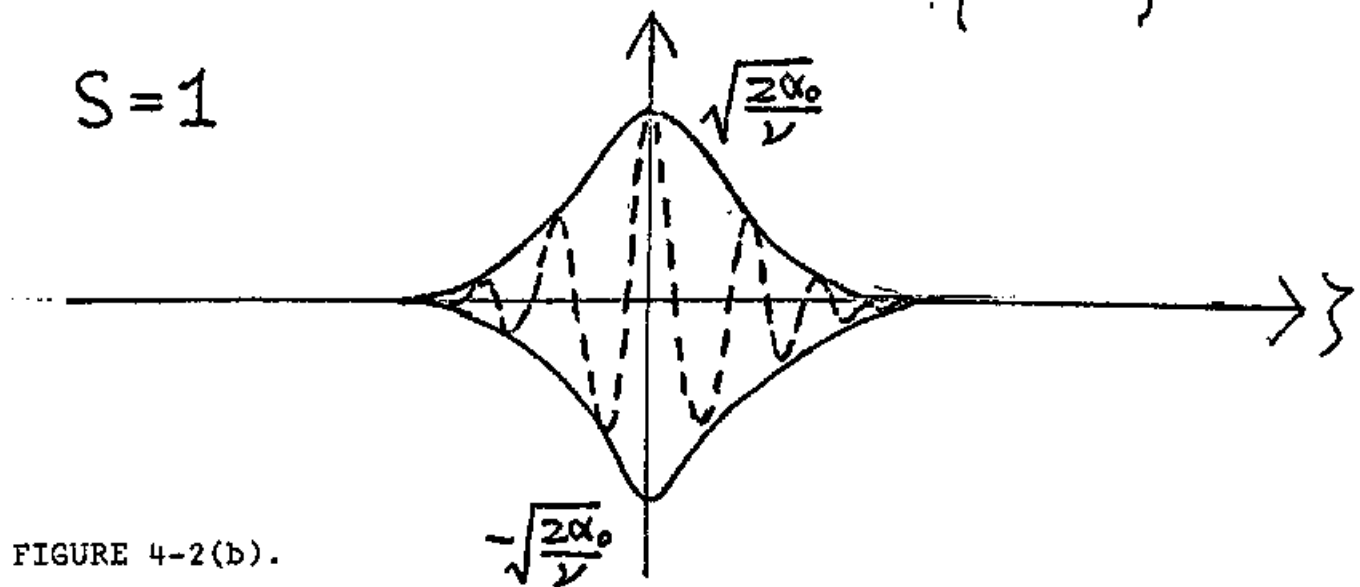


FIGURE 4-2(b).

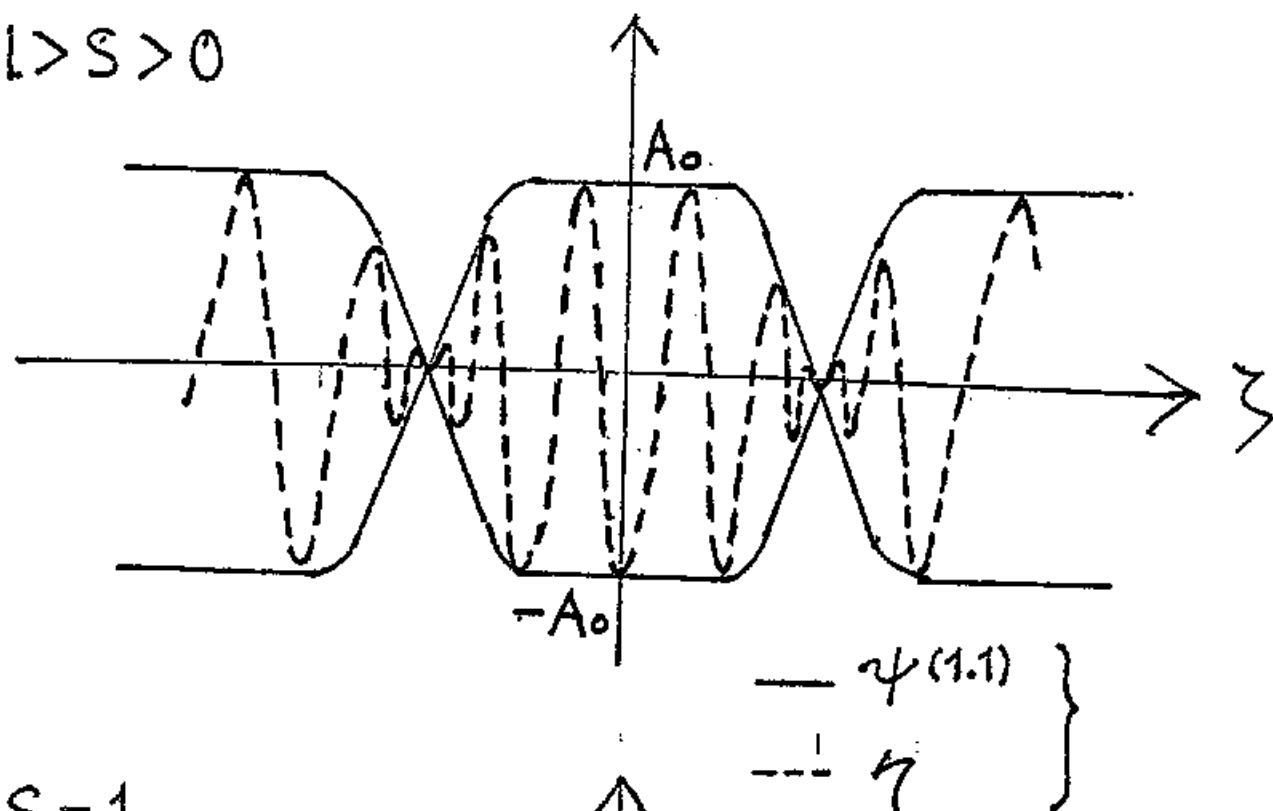
ENVELOPE SOLITON

FIGURE 4-3(a).

$$\zeta = \sqrt{-\frac{\nu}{2\mu}} \frac{A_0}{S} \zeta$$

PERIODIC ENVELOPE SPW ( $\mu\nu < 0$ )

$$1 > S > 0$$



$$S = 1$$

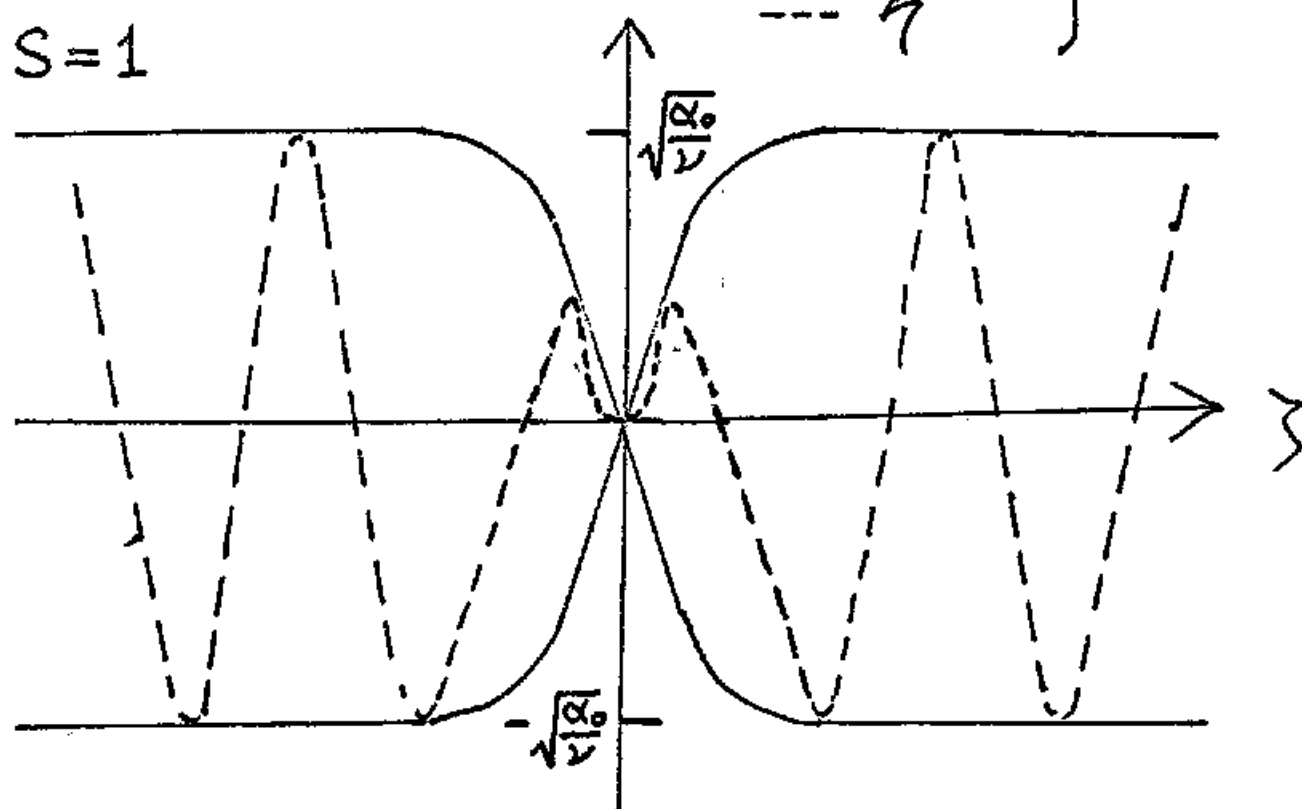


FIGURE 4-3(b).

PHASE GAP PROPAGATION

This solution, when  $s = 1$ , reduces to

$$\psi^{(1,1)} = \sqrt{\frac{\alpha_0}{\nu}} \tanh\left\{\sqrt{-\frac{\alpha_0 \nu}{2\mu}}\xi\right\} e^{i\alpha_0 \tau} \quad (4.31)$$

which represents the propagation of the phase gap( Fig.4-3(b)).

Chapter V                      Nonlinear Modulation of The Periodic SPW  
in Some Other Media

In the preceding chapter we showed that the nonlinear Schrodinger equation governs not only the Stokes waves on deep-water layer, but also the cnoidal waves on shallow-water layer. We note here that the nonlinear Schrodinger equation governing the modulation of the cnoidal waves can also be obtained directly from the K-dV equation, although we derived it from the original complicated nonlinear system of equation (4.1)-(4.4). Therefore it seems to be sufficient to treat the K-dV equation instead of the original system of equations, so far as the study of the modulation of weak cnoidal waves is concerned.

Extending this conjecture, we shall consider the nonlinear modulation of the periodic SPW in various nonlinear dispersive media which are approximately governed by the K-dV equation or the modified K-dV equation expressed as:

$$\frac{\partial u}{\partial t} + \alpha u^p \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0 \quad (p=1,2), \quad (5.1)$$

instead of starting from the original nonlinear system of equations. In the above equation,  $\alpha$  and  $\beta$  are parameters associated with each medium to be discussed ( see Table-II). We shall consider here the following three typical examples: (1) oblique magneto-acoustic waves<sup>32)</sup>, (2) Alfvén waves<sup>33)</sup> and (3) shallow-water waves including the effect of surface tension. It should be noted here that for the case of

magneto-acoustic waves,  $\beta$  changes its sign beyond a critical angle of propagation direction and also that for shallow-water waves,  $\beta$  takes negative sign when the effect of surface tension dominate over that of gravity force.

On the other hand, for the Alfvén waves  $\beta$  is always positive.

Applying a similar procedure to that adopted in the preceding chapter, that is, applying the coordinate-stretching together with the modified Fourier expansion to eq.(5.1), we have the following nonlinear Schrödinger equation for the complex amplitude  $\psi$  of the leading term in the expansion of  $u$ :

$$i \frac{\partial \psi}{\partial \tau} + \mu \frac{\partial^2 \psi}{\partial \xi^2} + \nu |\psi|^2 \psi = 0, \quad (5.2)$$

where

$$\begin{aligned} \mu &= -3k_0, \\ \nu &= \begin{cases} \alpha^2 / (6k_0 \beta) & \text{for } p=1 \\ -k_0 \alpha & \text{for } p=2 \end{cases} \end{aligned} \quad (5.3)$$

and  $k_0$  is the wave number of the linear wave.

The modulational stability of the cnoidal waves in the linearized limit is then determined by the criterion stated in §4-3 (Table-II). As easily seen from (5.3), the sign of  $\mu\nu$ , which characterizes the stability criterion, is independent of the sign of  $\beta$  since  $\mu\nu$  does not include  $\beta$  for  $p = 1$  where  $\beta$  can take both signs as noted before. Consequently it is found that the surface tension and the propagation direction do not affect, respectively, the stability criterion of the shallow-water waves and that of the magneto-acoustic waves.

TABLE-II

WAVE COEFFICIENT	MAGNETO-ACOUSTIC WAVE	ALFVÉN WAVE	SHALLOW-WATER WAVE
$\alpha$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3\sqrt{gh_0}}{2h_0}$
$\beta$	$\frac{1}{2} \left[ 1 - \left( \sqrt{\frac{M_i}{M_e}} - \sqrt{\frac{M_e}{M_i}} \right) \cot^2 \theta \right]$	$\frac{1}{2} \left[ 1 - \left( \sqrt{\frac{M_i}{M_e}} - \sqrt{\frac{M_e}{M_i}} \right) \cot^2 \theta \right]$	$\frac{3}{2} \sqrt{gh_0} \left[ \frac{1}{3} h_0^2 - \frac{T}{\rho g} \right]$
STABILITY	STABLE	UNSTABLE	STABLE
P	1	2	1

 $M_i$ : ION MASS $M_e$ : ELECTRON MASS $\theta$ : ANGLE BETWEEN EXTERNAL MAGNETIC FIELD AND PROPAGATION DIRECTION $h_0$ : DEPTH OF STILL WATERT: SURFACE TENSION  $\rho$ : WATER DENSITY

It is interesting to note that in two hydromagnetic modes one (magneto-acoustic mode ) is stable whereas the other ( Alfvén mode ) unstable. It should be noted that these results for the hydromagnetic waves coincide exactly with those obtained by Tam<sup>34)</sup>, who studied the stability of such waves by starting from the original complicated but exact system of equations. This agreement is quite remarkable, since our procedure is simply based upon the approximate equation instead of original system of equations.

For the complementary case to 'periodic'cnoidal waves considered here, Jeffrey and Kakutani<sup>35)</sup> showed, by the conventional stability theory, that the pulse like solitary wave solution to the K-dV equation is neutrally stable.



In the present thesis, we have considered the effects of inhomogeneity on the propagation of various kinds of SPW and also nonlinear modulation of SPW. We have shown, to the lowest order of approximation, that both the inhomogeneity effect and modulation can be governed by simple tractable equations such as (2.12), (3.15) or (4.9) for a considerably wide class of nonlinear waves. Although detailed analysis of these equations has not yet been made and remains as a future task, not a few results can be obtained by using these equations. We shall summarize and discuss here such analytical results so far obtained.

Firstly, it has been shown that a considerably wide class of nonlinear waves in inhomogeneous media can be described by a simple equation such as (2.12) and (3.15) which is of a similar form to the equation governing shallow-water waves over an uneven bottom. Thus interesting phenomena such as soliton fission may be expected to occur in other various nonlinear dispersive media. We have derived a formula giving the number of emergent solitons for a typical inhomogeneity. The result predicted by this formula shows qualitatively good agreement with the result obtained by numerical experiment for the shallow-water waves. This success may be due to the fact that the number of emergent solitons is determined only by the amplification effect of the

inhomogeneity but is independent of the asymmetric deformation of the incident soliton on the inhomogeneous region, which is easily observed in Fig.1-3 quoted from Madsen and Mei's paper. In order to clarify the detailed behaviour of the emergent solitons, however, we must take account of such asymmetry, which may be responsible to structure of oscillatory tail. It should be noted that although we have taken only a soliton as an incident wave, the applicability of the equation obtained here is not restricted to such a special situation.

It is also interesting to consider effect of inhomogeneity on E-soliton which was obtained in Chapt.IV as a SPW solution of the nonlinear Schrodinger equation. It is highly probable that E-soliton may split into a number of solitons in inhomogeneous media. But this requires a separate study.

Secondly, we have shown that the nonlinear modulation of the periodic SPW on the surface of water layer can be described by the nonlinear Schrödinger equation. This type of equation has also been obtained in the studies of other nonlinear dispersive media. Therefore the results obtained in this thesis for the water waves can easily be extended to other dispersive waves if we make an appropriate interpretation. For the water waves, we have shown that an infinitesimal modulation of periodic SPW on deep- water layer (i.e. Stokes wave ) leads to finite deformation and it is highly probable that the deformed SPW may degenerate into a new type of SPW (i.e. E-soliton ). On the other hand, the periodic SPW on shallow-water layer (i.e. cnoidal wave ) is found to be stable for

small modulation but it may be deformed into another SPW such as phase gap for finite modulation.

Concerning the instability of the basic SPW on deep-water layer, it is interesting to consider the second instability of the new SPW resulting from the basic one. So far as E-soliton is concerned, many numerical experiments suggest its stability<sup>27,30,31</sup>). On the other hand, for the envelope periodic SPW, its stability is not necessarily guaranteed as stated in INTRODUCTION. If the envelope periodic SPW is again unstable, the successive instability process will require a statistical treatment of wave motion. In connection with this, it is interesting to note that a generalized nonlinear Schrodinger equation, whose coefficients are complex, has been obtained in the field of the nonlinear stability theory<sup>35</sup>), which is concerned with the problem seeking for the missing link between the laminar state and turbulent one of fluid motion.

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